§4.3 Counterterms and Physical Perturbation Theory

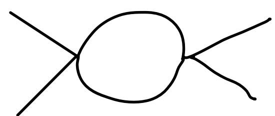
Mass renormalization:

Consider again the φ^4-theory

$\to$ λ_p is function of s_0, t_0, u_0

set them all equal to μ^2

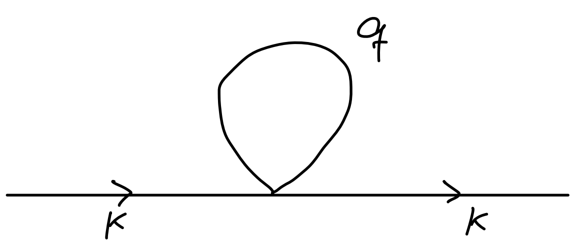
$$\to \quad -i\lambda_p = -i\lambda + 3iC\lambda^2 \log\left(\frac{\Lambda^2}{\mu^2}\right) + O(\lambda^3)$$

Inserting into amplitude 

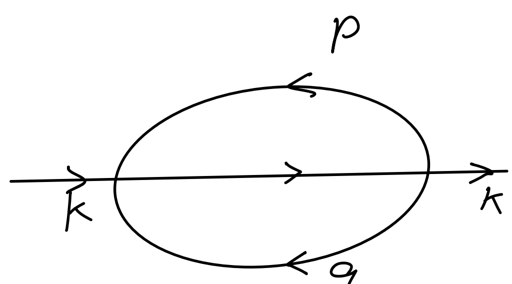
shows that dependence on Λ disappears

Question: But does this hold for all Feynman diagrams?

Consider corrections to the φ-propagator:



(a)



(b)

Diagram (a) gives

$$-i\lambda \int^{\Lambda} \frac{d^4q}{(2\pi)^4} \frac{i}{q^2 - m^2 + i\varepsilon}$$

$\rightarrow$ depends quadratically on Λ but not on k^2!

Diagram (b) involves double integral

$$I(k, m, \Lambda; \lambda) = (-i\lambda)^2 \int^{\Lambda}\int^{\Lambda} \frac{d^4p}{(2\pi)^4} \frac{d^4q}{(2\pi)^4}$$

$$\times \frac{i}{p^2 - m^2 + i\varepsilon} \frac{i}{q^2 - m^2 + i\varepsilon} \frac{i}{(p+q+k)^2 - m^2 + i\varepsilon}$$

counting powers of p and q gives

$$I \sim \int^{\Lambda} \left(\frac{d^8P}{P^6} \right) \sim \text{quadratic in } \Lambda$$

By Lorentz invariance, I is function of k^2 and we can write:

$$I = D + E k^2 + F k^4 + \dots$$

$$\rightarrow D \sim \Lambda^2, \quad E = \frac{1}{2} \frac{d^2 I(k)}{dk^2}\bigg|_{k=0} \sim \int \frac{d^8P}{P^8} \sim \log \Lambda$$

$$F \sim \frac{d^4 I(k)}{dk^4} \sim \int \frac{d^8P}{P^{10}} \rightarrow \text{convergent !}$$

So what is all this imply ?

Recall the definition of Green functions:

$$G^{(N)}(k_1, \ldots, k_N) = \left. \frac{\delta^N Z[J]}{\delta J(k_1) \ \cdots \ \delta J(k_N)} \right|_{J=0} \quad (1)$$

where $Z[J]$ is the partition function with source J coupled to φ with vaccuum contribution Z_0 removed

$$\rightarrow Z[J] = 1 + \sum_{N=1}^{\infty} \frac{1}{N!} \sum_{k_1 \cdots k_N} G^{(N)}(k_1, \ldots, k_N) J(k_1) \cdots J(k_N) \quad (2)$$

Now define functional $W[J]$ via

$$i W[J] = \sum_{N=1}^{\infty} \frac{1}{N!} \sum_{k_1 \cdots k_N} G_c^{(N)}(k_1, \ldots, k_N) J(k_1) \cdots J(k_N) \quad (3)$$

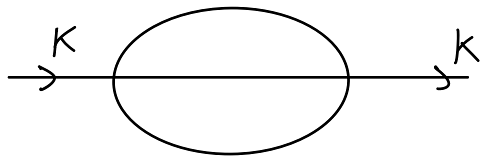
where $G_c^{(N)}$ are the connected parts of $G^{(N)}$'s

$$\rightarrow \quad Z[J] = e^{iW[J]} \quad (4)$$

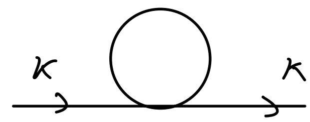
and we have

$$G_c^{(N)}(k_1, \ldots, k_N) = i \left. \frac{\delta^N W[J]}{\delta J(k_1) \ \cdots \ \delta J(k_N)} \right|_{J=0} \quad (5)$$

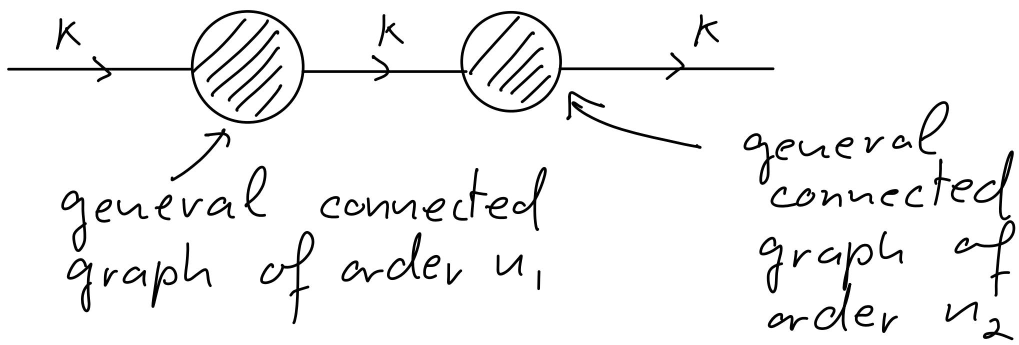
Let us now come back to the corrections to our φ-propagator:



and



More generally, we can have corrections of the form



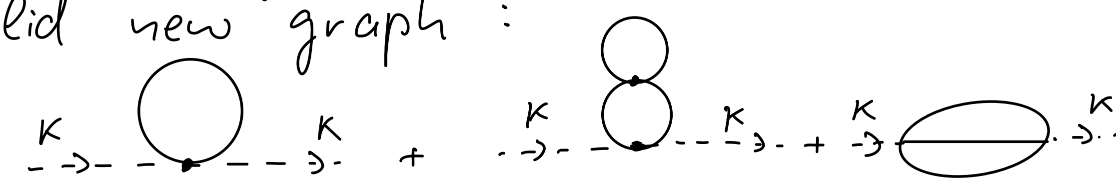
order of total graph: $n_1 + n_2$

$\rightarrow$ contributes to partition function (2) with factor $[(n_1+n_2)!]^{-1}$

but in the bulbs we have factors $(n_1!)^{-1}$ and $(n_2!)^{-1}$

$\rightarrow$ we also have to account for the number of ways the $n_1 + n_2$ vertices can be distributed among the two bulbs giving $[(n_1+n_2)!]^{-1} \dfrac{(n_1+n_2)!}{n_1!\, n_2!} = [n_1!\, n_2!]^{-1}$

Define now $\Sigma(k)$ as the sum of all graphs of $G_c^{(2)}(k)$ which are "one-particle irreducible", that is when cutting a line does <u>not</u> produce a valid new graph:

$\Sigma =$ 

$+ \; \mathcal{O}(\lambda^3)$

where dotted propagators are removed

$$\longrightarrow \quad G_c^{(2)}(k) = \underbrace{G_0^{(2)}(k)}_{\text{propagator}} + G_0(k)\Sigma(k)G_0^{(2)}(k)$$

$$+ \; G_0^{(2)}(k)\,\Sigma(k)\,G_0^{(2)}(k)\,\Sigma(k)\,G_0^{(2)}(k) + \dots$$

$$= G_0^{(2)}(k)\left(1 + \left(\Sigma(k)\,G_0(k)\right) + \left(\Sigma G_0\right)\left(\Sigma G_0\right) + \dots\right)$$

$$= G_0^{(2)}(k)\left[1 - \Sigma(k)G_0(k)\right]^{-1}$$

$$= \left[G_0^{-1}(k) - \Sigma(k)\right]^{-1} \qquad (6)$$

Putting it all together, we get from our diagrams (a) and (b): $\Sigma(k) = a + bk^2 + \mathcal{O}(k^4)$ with $a \sim \Lambda^2$ and $b \sim \log \Lambda$

$$(G) \longrightarrow \frac{1}{k^2 - m^2} \longmapsto \frac{1}{(1-b)k^2 - (m^2+a)}$$

$\rightarrow$ the pole in k^2 is shifted to

$$m_p^2 := m^2 + \delta m^2 = (m^2+a)(1-b)^{-1}$$

physical mass

"mass renormalization"

We also notice that the residue of the pole changed to $(1-b)^{-1}$

<u>interpretation</u>: the field φ is normalized such that $\mathcal{L} = \frac{1}{2}(\partial\varphi)^2 + \ldots$

change of residue implies that quantum corrections modify $\mathcal{L}$ to

$$\mathcal{L}' = \frac{1}{2}(1-b)^{-1}(\partial\varphi)^2 + \ldots$$

$\rightarrow$ renormalize φ to $\varphi' = (1-b)^{-1/2}\varphi$

"field renormalization"

Bare versus physical perturbation theory

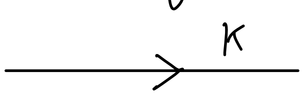
Putting subscripts 0 on our Lagrangian fields/couplings: φ_0, m_0, λ_0,
we see that they are the "bare" field, and m_0, λ_0 are bare mass and coupling const.

Bare quantities can be cutoff-dependent and divergent!

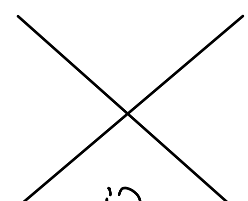
Instead use notation:

$$\mathcal{L} = \frac{1}{2}\left[(\partial\varphi)^2 - m_p^2\varphi^2\right] - \frac{\lambda_p}{4!}\varphi^4 + A(\partial\varphi)^2 + B\varphi^2 + C\varphi^4$$

$\longrightarrow$ Feynman rules:

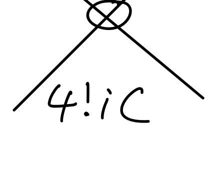


$\dfrac{i}{k^2 - m_p^2}$



$-i\lambda_p$

$+2i(Ak^2 + B)$



$4!\,iC$

A, B and C are determined iteratively:

to order λ_p^N we write A_N, B_N and C_N

→ draw Feynman diagrams to order λ_p^{N+1}

→ determine A_{N+1}, B_{N+1}, and C_{N+1} by requiring that

i) the propagator calculated to order λ_p^{N+1} has pole at $k^2 = m_p^2$

ii) with residue $= 1$,

iii) and that the φ-φ scattering amplitude at some prescribed kinematical variables has value $-i\lambda_p$

→ 3 conditions for fixing A_{N+1}, B_{N+1} and C_{N+1} !

<u>The only way</u> this can go wrong is if perturbation theory produces some diagram with 6 (or more) external legs that is cutoff-dep.

$\longrightarrow$ there is no $D\varphi^6$ counterterm and we are in trouble!

Let's see why (and when) this is avoided

Degree of divergence

Consider a diagram with E external φ lines.

Definition: A diagram is said to have a "superficial degree of divergence" D if it diverges as Λ^D

Theorem: $D = 4 - E$

check: for
$E = 2 \longrightarrow D = 2$
$E = 4 \longrightarrow D = 0$ (logarithmic)
$E = 6 \longrightarrow D = -2$ (convergent)

we don't have to worry!